A new approach to the asymptotics for Sobolev orthogonal polynomials

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Abstract

In this paper we deal with polynomials orthogonal with respect to an inner product involving derivatives, that is, a Sobolev inner product. Indeed, we consider Sobolev type polynomials which are orthogonal with respect to

$$(f,g)=\int fgd\mu+\sum_{i=0}^{r}M_i f^{(i)}(0)g^{(i)}(0), \quad M_i\geq 0,$$

where μ is a certain probability measure with unbounded support. For these polynomials, we obtain the relative asymptotics with respect to orthogonal polynomials related to μ, Mehler–Heine type asymptotics and their consequences about the asymptotic behaviour of the zeros.

To establish these results we use a new approach different from the methods used in the literature up to now. The development of this technique is highly motivated by the fact that the methods used when μ is bounded do not work.

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1 Introduction

Let $\{\mu_i\}_{i=0}^r$ be Borel positive measures supported on the real line. We can define the Sobolev space:

$$W^{2,r}(\mu_0, \mu_1, \ldots, \mu_r) := \{f : \int |f|^2 d\mu_0 + \sum_{i=1}^{r} \int |f^{(i)}|^2 d\mu_i < +\infty\}$$

with the inner product

$$(f, g) = \int f\, g d\mu_0 + \sum_{i=1}^{r} \int f^{(i)}\, g^{(i)} d\mu_i.$$

It is very well known that this inner product is nonstandard, that is, $(xf, g) \neq (f, xg)$. Therefore, the nice properties of the standard orthogonal polynomials, such as the three–term recurrence relation, the interlacing properties of the zeros, etc, do not hold any more. Then, the powerful methods and techniques developed for over a century to study *standard* orthogonal polynomials could not work (in fact, they cannot work) for Sobolev orthogonal polynomials. Then, a question arises: *is it necessary or interesting to study these "pathological" polynomials?* In our opinion the answer is affirmative. It does not exist a general theory for these families of orthogonal polynomials, for example, up to now powerful tools such as Rieman–Hilbert approach to obtain asymptotic properties of these polynomials have not worked. This should be a motivation to pay attention to Sobolev orthogonal polynomials, that is, to investigate how to construct a more general theory in the same sense as it was made for the standard orthogonal polynomials many years ago. Furthermore, some applications of the Sobolev orthogonality in the theory of standard orthogonal polynomials are known, for instance, standard polynomials with nonstandard parameters are not orthogonal in the usual sense but they are orthogonal with respect to Sobolev inner products (see among others [1], [7] or [12]).

Thus, according to the above reasoning, in this paper we take a step to get a better knowledge of the properties of the Sobolev orthogonal polynomials, more concretely of the discrete Sobolev orthogonal polynomials. Let μ be a finite positive Borel measure supported on the real line, $c \in \mathbb{R}$ and $M_i \geq 0$ for $i = 0, 1, ..., r$. We consider an inner product of the form

$$(f, g) = \int f(x)g(x)d\mu(x) + \sum_{i=0}^{r} M_i f^{(i)}(c)g^{(i)}(c),$$

and let $\{Q_n\}_{n\geq 0}$ be the corresponding sequence of monic orthogonal polynomials. More general products where cross–product terms appear in the discrete part have also been studied. But, recently in [13] the authors prove that every symmetric bilinear form can be reduced to a diagonal case, that is, without cross–product terms.

The aim is to compare the Sobolev orthogonal polynomials with the standard orthogonal polynomials associated with the measure μ in order to investigate how the addition of the derivatives in the inner product influences the orthogonal system.

There exist many formal results for these polynomials: recurrence relation, location of zeros, differential formulas, and so on. However, little is known concerning the asymptotic properties. We want to remark that most of the general results have been obtained when $\operatorname{supp}(\mu)$ is a bounded set. More precisely, in [14], the authors assume that μ is a measure for which the asymptotic behaviour of the orthogonal polynomials is known; the most relevant class of this type is the Nevai class $M(0,1)$ of orthogonal polynomials with appropriately converging recurrence coefficients. They studied the relative asymptotics when the mass point c is outside the support of the measure. The same product with the mass point in $\operatorname{supp}(\mu)$ has been studied in [17].

In both papers, the key is the possibility to transform the Sobolev orthogonality into the standard quasi–orthogonality. As a consequence, we can express the polynomial Q_n as a linear combination (with a fixed number of terms) of standard orthogonal polynomials R_n corresponding to the modified measure $d\nu = (x-c)^{r+1}d\mu$, that is,

$$Q_n(x) = \sum_{j=0}^{r+1} a_n^j R_{n-j}(x)\,. \tag{1}$$

We want to point out that in the bounded case, a straightforward argument yields to prove that all the connexion coefficients a_n^j are bounded. This behaviour of the coefficients together with the fact that the orthogonal polynomials R_n have an adequate finite ratio asymptotics is enough reason to study each term of (1) separately, in order to get the relative asymptotics for Q_n (see [14] and [17] where this technique is developed).

However, the situation is quite different if we deal with the unbounded case. More concretely, we consider the Laguerre probability measure, that

is, $d\mu(x) = \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}dx$ with $\alpha > -1$, and the inner product

$$(f,g)_r = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(x)g(x)x^\alpha e^{-x}dx + \sum_{i=0}^{r} M_i f^{(i)}(0)g^{(i)}(0), \qquad (2)$$

where $M_i > 0$, $i = 0, \ldots, r$. Then we will see in Theorem 1 that the connexion coefficients which appear in (1) are unbounded. So, as we will see later, when we try to obtain the relative asymptotics with the techniques used for the bounded case and we take into account the ratio asymptotics for Laguerre polynomials, we come across a serious problem. Indeed, we find that each term of (1) tends to infinity, all of them being the same order, but with an alternating sign. Then, the idea that each term has a limit does not work now, and therefore we can say that there is an intrinsic difficulty related to the unbounded case.

The interest lies in knowing the differences between the Laguerre polynomials and the Sobolev polynomials $Q_{n,r}$ orthogonal with respect to (2). Intuitively one can imagine that the asymptotic differences in the complex plane should be around the perturbation of the standard inner product involved in the Sobolev inner product, that is, around the origin. To confirm this, first we get the relative asymptotics in Theorem 2 and we prove that both families of orthogonal polynomials $Q_{n,r}$ and L_n^α are identical asymptotically on compact subsets of $\mathbb{C} \setminus [0, \infty)$.

Later, we consider Mehler–Heine type formulas because they are nice tools to describe the Laguerre–Sobolev type polynomials around the origin. In [5], with $r = 1$ and $M_0, M_1 > 0$ the authors find a behaviour pattern and they establish a conjecture which is reformulated in [15] more clearly than in [5]. That is:

If $M_i > 0$ for $i = 0, \ldots, r$, in the inner product (2), then

$$\lim_{n\to\infty} \frac{(-1)^n}{n!n^\alpha} Q_{n,r}\left(\frac{x}{n+j}\right) = (-1)^{r+1} x^{-\alpha/2} J_{\alpha+2r+2}(2\sqrt{x}),$$

uniformly on compact subsets of the complex plane and on $j \in \mathbb{N} \cup \{0\}$ where $\{Q_{n,r}\}_{n\geq 0}$ is the sequence of monic orthogonal polynomials with respect to (2) and J_α is the Bessel function of the first kind of order α.

In Theorem 3 we prove that this conjecture is true and this is one of our main results. We would like to note that the techniques used to prove it are not a simple generalization of the ones used in [5]. There, the authors consider the algebraic expression

$$Q_{n,1}(x) = B_0(n)L_n^{\alpha}(x) + B_1(n)xL_{n-1}^{\alpha+2}(x) + B_2(n)x^2L_{n-2}^{\alpha+4}(x)$$

where the coefficients $B_i(n)$ were given explicitly in [11]. Now, in a discrete Laguerre–Sobolev inner product with an arbitrary number of terms, the problem is that we only have an algebraic expression given in [10], but not the explicit expression of the coefficients $B_i(n)$, of which we only know that they are a non trivial solution of a system with $r+1$ equations and $r+2$ unknowns.

The approach used to prove Theorem 3 is totally new. We do not use algebraic tools but analytic ones. For this purpose, we obtain a new and nice formula for the derivatives of $Q_{n,r}$ and as a consequence, we achieve a uniform bound for the ratios $\frac{Q_{n,r}^{(k)}(0)}{(L_n^{\alpha})^{(k)}(0)}$.

These Mehler–Heine type formulas are interesting twofold: they provide the scaled asymptotics for $Q_{n,r}$ on compact sets of the complex plane and they supply us with asymptotic information about the location of the zeros of these polynomials in terms of the zeros of other known special functions. More precisely, applying Hurwitz's Theorem in a straightforward way, we prove that there exists an acceleration of the convergence of $r+1$ zeros of these Sobolev polynomials towards the origin.

Along the paper, we also deal with the possibility of some $M_i = 0$. We call such Sobolev inner product, a Sobolev inner product with *holes*. More concretely, we consider the inner product

$$\begin{aligned}&(f,g)_{r,s}\\ &= \frac{1}{\Gamma(\alpha+1)}\int_0^{\infty} f(x)g(x)x^{\alpha}e^{-x}dx + \sum_{i=0}^{r} M_i f^{(i)}(0)g^{(i)}(0) + M_s f^{(s)}(0)g^{(s)}(0),\end{aligned}$$

where $s \geq r+2$ and $M_i > 0$ for $i = 0, \ldots, r$ and $i = s$.

For this situation, we also establish the relative asymptotics and the Mehler-Heine type formulas for these orthogonal polynomials. We want to remark that this case has qualitative differences with respect to the case without holes. For example, concerning the convergence acceleration to 0 of the zeros of the polynomials, the result does not depend on the number of terms in the discrete part, but it depends on the position of the hole. So, despite the presence of the mass M_s, there only exists an acceleration of the convergence of $r+1$ zeros such as it occurs in the case of the inner product without holes.

As a consequence of all the previous results, using the symmetrization process in the framework of Sobolev type orthogonal polynomials, we can prove in Proposition 1 the relative asymptotics and the Mehler–Heine type formulas for the generalized Hermite-Sobolev type polynomials. Furthermore, we hope this method can be used with other unbounded measures for which we have quite less explicit information about the corresponding orthogonal polynomials.

The structure of the paper is as follows. In Section 2 we introduce the notation, the basic tools, and some properties of classical Laguerre polynomials. In Section 3 we obtain a new auxiliary result in Laguerre–Sobolev type orthogonal polynomials that we use to establish our main results in Sections 4 and 5. Concretely, Section 4 is devoted to obtain the relative asymptotics and the Mehler–Heine type formula for the orthogonal polynomials with respect to a discrete Sobolev inner product with positive masses and in Section 5 we get the corresponding ones for the orthogonal polynomials with respect to an inner product with holes. We remark that exterior strong asymptotics for the sequences of Sobolev orthogonal polynomials considered in Sections 4 and 5 are trivially deduced from the relative asymptotics obtained here, because we know explicitly the exterior strong asymptotics for the corresponding sequences of orthogonal polynomials with respect to μ. Furthermore, exterior Plancherel–Rotach asymptotics for these Sobolev orthogonal polynomials can also be obtained in a similar way with the technique introduced here.

2 Notation and basic results

Along this work we will deal with classical Laguerre polynomials, that is, polynomials orthogonal with respect to the inner product in the space of all polynomials with real coefficients

$$(p,q) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty p(x)q(x)\, x^\alpha e^{-x}\, dx\,, \quad \alpha > -1.$$

We will denote by L_n^α the nth monic Laguerre polynomial.

Many of the properties of Laguerre polynomials can be seen, for example, in Szegő's book [18]. In what follows we summarize those properties which will be used in this paper.

It is known that the *monic Laguerre polynomials* are defined by

$$L_n^\alpha(x) = (-1)^n n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k} (-1)^k \frac{x^k}{k!}\,,$$

and their L_2-norm is

$$\|L_n^\alpha\|^2 = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty (L_n^\alpha(x))^2\, x^\alpha\, e^{-x}\, dx = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} n!\,. \qquad (3)$$

The evaluation at $x=0$ of the polynomial L_n^α and its successive derivatives are given by

$$(L_n^\alpha)^{(k)}(0) = \frac{(-1)^k\, n!}{(n-k)!} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)} L_n^\alpha(0) = \frac{(-1)^{n+k}\, n!}{(n-k)!} \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+k+1)}. \quad (4)$$

A useful tool to some estimates is the *Stirling formula*:

$$\Gamma(x+1) \sim x^x e^{-x} \sqrt{2\pi x} \quad (x \to +\infty)$$

where the symbol $f(x) \sim g(x) \quad (x \to a)$ stands for $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$. In particular,

$$\Gamma(n+\alpha+1) \sim n!\, n^\alpha. \qquad (5)$$

As a consequence, from (3), (4), and (5), we get

$$\lim_n \frac{(L_n^\alpha(0))^2}{\|L_n^\alpha\|^2\, n^\alpha} = \frac{1}{\Gamma(\alpha+1)}\,. \qquad (6)$$

The following asymptotic results are known. They can be deduced from Perron's formula in Szegő's book [18],

$$\lim_n \frac{n L_{n-1}^\alpha(x)}{L_n^\alpha(x)} = -1, \qquad (7)$$

$$\lim_n \frac{n^{1/2} L_n^\alpha(x)}{L_n^{\alpha+1}(x)} = \sqrt{-x}, \qquad (8)$$

both uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$.

The nth kernel for the Laguerre polynomials $K_n(x,y) = \sum_{i=0}^{n} \frac{L_i^\alpha(x) L_i^\alpha(y)}{\|L_i^\alpha\|^2}$ satisfies the Christoffel–Darboux formula

$$K_n(x,y) = \frac{1}{\|L_n^\alpha\|^2} \frac{L_{n+1}^\alpha(x) L_n^\alpha(y) - L_{n+1}^\alpha(y) L_n^\alpha(x)}{x-y}\,.$$

As usual, we denote the derivatives of the kernels by

$$K_n^{(k,s)}(x,y) = \frac{\partial^{k+s}}{\partial x^k \partial y^s} K_n(x,y) = \sum_{i=0}^{n} \frac{(L_i^\alpha)^{(k)}(x)(L_i^\alpha)^{(s)}(y)}{\|L_i^\alpha\|^2}$$

with $k, s \in \mathbb{N} \cup \{0\}$ and the convention $K_n^{(0,0)}(x,y) = K_n(x,y)$.

In the next lemma we show some formulas for the derivatives of the kernels that we will need throughout the paper.

Lemma 1 *The derivatives of the kernels of the Laguerre polynomials, for $k, s \in \mathbb{N} \cup \{0\}$, satisfy*

(a)

$$K_{n-1}^{(0,s)}(x,0) = \frac{1}{\|L_{n-1}^\alpha\|^2} \frac{s!}{x^{s+1}} \left[P_s(x,0;L_{n-1}^\alpha) L_n^\alpha(x) - P_s(x,0;L_n^\alpha) L_{n-1}^\alpha(x) \right]$$

where $P_s(x,0;f)$ is the sth Taylor polynomial of f at 0.

(b)

$$K_{n-1}^{(k,s)}(0,0) = \frac{k!\,s!}{\|L_{n-1}^\alpha\|^2} \sum_{j=0}^{s} \frac{k+s+1-2j}{n-j} \, \frac{(L_{n-1}^\alpha)^{(j)}(0)(L_n^\alpha)^{(k+s+1-j)}(0)}{j!\,(k+s+1-j)!},$$

$$K_{n-1}^{(k,0)}(0,0) = (-1)^k \frac{\Gamma(\alpha+n+1)}{(n-(k+1))!\,\Gamma(\alpha+k+2)}.$$

Proof. (a) The result follows from the Christoffel-Darboux formula and Leibniz's rule.

(b) Observe that, according to Taylor's formula, $\frac{1}{k!} K_{n-1}^{(k,s)}(0,0)$ is precisely the coefficient of x^k in $K_{n-1}^{(0,s)}(x,0)$, therefore

$$K_{n-1}^{(k,s)}(0,0)$$
$$= \frac{k!\,s!}{\|L_{n-1}^\alpha\|^2} \sum_{j=0}^{s} \frac{(L_{n-1}^\alpha)^{(j)}(0)(L_n^\alpha)^{(k+s+1-j)}(0) - (L_n^\alpha)^{(j)}(0)(L_{n-1}^\alpha)^{(k+s+1-j)}(0)}{j!\,(k+s+1-j)!}.$$

In particular for $s=0$, straightforward computations lead us to conclude this lemma. $\square$

Along the paper we work with sequences of monic orthogonal polynomials, and we use the acronym SMOP for them.

3 Auxiliary results

From now on $\{Q_{n,r}\}_{n\geq 0}$ denotes the sequence of monic Laguerre–Sobolev orthogonal polynomials with respect to an inner product of the form

$$(p,q)_r = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty p(x)q(x)\, x^\alpha e^{-x}\, dx + \sum_{i=0}^{r} M_i p^{(i)}(0)q^{(i)}(0), \qquad (9)$$

where $\alpha > -1$ and $M_i > 0$, $i = 0, \ldots, r$. Notice that all the masses in the discrete part of this inner product are positive.

We write $K_{n,r}$ for the corresponding nth kernel, that is $K_{n,r}(x,y) = \sum_{j=0}^{n} \dfrac{Q_{j,r}(x)Q_{j,r}(y)}{(Q_{j,r}, Q_{j,r})_r}$, and $K_{n,r}^{(k,s)}$ for the derivatives of the kernels.

Observe that, in fact, $(.,.)_r$, $Q_{n,r}$, $K_{n,r}$ and $K_{n,r}^{(k,s)}$ also depend on the parameter α but for simplicity we have omitted it in the notations.

In the next lemma, we obtain an asymptotic estimation for $Q_{n,r}^{(k)}(0)$, $k \geq 0$, that will play an important role along this paper. To do this, we need to know the "size" of the kernels of $Q_{n,r}$ and their derivatives. The following discrete version of l'Hospital's rule (see e.g. [9]) will be very helpful to easily calculate some limits:

Stolz Criterion. Let $\{x_n\}$ and $\{y_n\}$ be real sequences. Suppose that $\{y_n\}$ is monotonic and $y_n \neq 0$ for all n. If $\lim_n (x_{n+1} - x_n)/(y_{n+1} - y_n) = L \in \mathbb{R}\cup\{\pm\infty\}$ exists, then $\lim_n x_n/y_n = L$ provided either $\lim_n x_n = \lim_n y_n = 0$ or $\lim_n y_n = \pm\infty$.

Lemma 2 *Let $Q_{n,r}$ be the monic polynomials orthogonal with respect to the inner product (9). Then the following statements hold:*

(a) *For $0 \leq k \leq r$,*

$$\frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} \sim \frac{C_{r,k}}{n^{\alpha+2k+1}},$$

where $C_{r,k}$ is a nonzero real number independent of n.

For $k \geq r+1$,

$$\lim_n \frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} = \frac{k!}{(k-(r+1))!}\,\frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+r+k+2)}\,.$$

(b)

$$\lim_n \frac{(Q_{n,r}, Q_{n,r})_r}{\|L_n^\alpha\|^2} = 1\,.$$

Proof. We use mathematical induction on $r \in \mathbb{N} \cup \{0\}$.

If $r = 0$, the Fourier expansion of the polynomial $Q_{n,0}$ in the orthogonal basis $\{L_n^\alpha\}_{n\geq 0}$ leads to

$$Q_{n,0}(x) = L_n^\alpha(x) - M_0 Q_{n,0}(0) K_{n-1}(x,0)\,,$$

and therefore

$$Q_{n,0}(x) = L_n^\alpha(x) - \frac{M_0 L_n^\alpha(0)}{1 + M_0 K_{n-1}(0,0)} K_{n-1}(x,0)\,. \tag{10}$$

As a consequence of (4) and Lemma 1 (b), we obtain (a) for $r = 0$.
Using (10), we have

$$(Q_{n,0}, Q_{n,0})_0 = \|L_n^\alpha\|^2 + \frac{M_0 (L_n^\alpha(0))^2}{1 + M_0 K_{n-1}(0,0)}\,.$$

Thus, from (6) and Lemma 1 (b) it follows (b) for $r = 0$.

Suppose now that (a) and (b) hold for the SMOP $\{Q_{n,r}\}_{n\geq 0}$ with $r > 0$, then we are going to deduce that they are also true for the sequence $\{Q_{n,r+1}\}_{n\geq 0}$. To do this, we observe that

$$(p,q)_{r+1} = (p,q)_r + M_{r+1} p^{(r+1)}(0) q^{(r+1)}(0) \tag{11}$$

and therefore

$$Q_{n,r+1}(x) = Q_{n,r}(x) - M_{r+1} Q_{n,r+1}^{(r+1)}(0) K_{n-1,r}^{(0,r+1)}(x,0)\,. \tag{12}$$

Taking derivatives $r+1$ times in (12) and evaluating at $x = 0$, we obtain

$$Q_{n,r+1}^{(r+1)}(0) = \frac{Q_{n,r}^{(r+1)}(0)}{1 + M_{r+1} K_{n-1,r}^{(r+1,r+1)}(0,0)}\,. \tag{13}$$

Taking now derivatives k times in (12), evaluating at $x = 0$, and using (13) we get

$$\frac{Q_{n,r+1}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} = \frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} - \frac{M_{r+1} K_{n-1,r}^{(k,r+1)}(0,0)}{1 + M_{r+1} K_{n-1,r}^{(r+1,r+1)}(0,0)} \frac{Q_{n,r}^{(r+1)}(0)}{(L_n^\alpha)^{(k)}(0)}\,. \tag{14}$$

Before taking limits in the last expression, we need to estimate $K_{n-1,r}^{(k,r+1)}(0,0)$. Applying Stolz criterion, the induction hypothesis for $\{Q_{n,r}\}_{n\geq 0}$, (4) and (6), we obtain for $k \geq r+1$

$$
\begin{aligned}
\lim_n \frac{K_{n-1,r}^{(k,r+1)}(0,0)}{n^{\alpha+k+r+2}} &= \lim_n \frac{Q_{n-1,r}^{(k)}(0)Q_{n-1,r}^{(r+1)}(0)}{\|L_{n-1}^{\alpha}\|^2(\alpha+k+r+2)\; n^{\alpha+k+r+1}} \\
&= \frac{(-1)^{k+r+1}\,\Gamma(\alpha+1)}{(\alpha+k+r+2)\Gamma(\alpha+k+1)\Gamma(\alpha+r+2)} \lim_n \left[\frac{Q_{n-1,r}^{(k)}(0)}{(L_{n-1}^{\alpha})^{(k)}(0)} \; \frac{Q_{n-1,r}^{(r+1)}(0)}{(L_{n-1}^{\alpha})^{(r+1)}(0)} \right] \\
&= \frac{k!(r+1)!}{(k-(r+1))!} \frac{(-1)^{k+r+1}\Gamma(\alpha+1)}{\Gamma(\alpha+k+r+3)\Gamma(\alpha+2r+3)}\,, \qquad (15)
\end{aligned}
$$

and therefore, from (14), we get (a) for $k \geq r+1$.

Now, if $0 \leq k \leq r$, to estimate the size of $K_{n-1,r}^{(k,r+1)}(0,0)$, we use Stolz criterion again and thus, we obtain

$$
\lim_n \frac{K_{n-1,r}^{(k,r+1)}(0,0)}{n^{r+1-k}} = (-1)^{r+1+k} \frac{(r+1)!}{r+1-k} \frac{C_{r,k}\,\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)\Gamma(\alpha+2r+3)}\,.
$$

Using the induction hypothesis and substituting all these results in the right–hand side of (14) we get

$$
\frac{Q_{n,r+1}^{(k)}(0)}{(L_n^{\alpha})^{(k)}(0)} \sim \frac{C_{r+1,k}}{n^{\alpha+2k+1}}\,,
$$

where $C_{r+1,k} = -\frac{\alpha+k+r+2}{r+1-k} C_{r,k} \neq 0$. Therefore the proof of (a) is complete.

To finish the proof of the Lemma, we only need to prove (b) for $\{Q_{n,r+1}\}_{n\geq 0}$. As in the case $r=0$, from (11) and (13), we get

$$
(Q_{n,r+1}, Q_{n,r+1})_{r+1} = (Q_{n,r}, Q_{n,r})_r + \frac{M_{r+1}(Q_{n,r}^{(r+1)}(0))^2}{1+M_{r+1}K_{n-1,r}^{(r+1,r+1)}(0,0)}\,.
$$

Using (a) for $k=r+1$, (3), (4) and (15) we achieve the result. $\square$

4 Main results

As we have mentioned in the introduction, if we consider a general discrete Sobolev inner product where the support of the measure μ is a bounded set, the key used to obtain some results is the possibility to transform the Sobolev orthogonality into a standard quasi–orthogonality.

Now, in our particular case, the sequence $\{Q_{n,r}\}_{n\geq 0}$ orthogonal with respect to the inner product defined by (9) is quasi–orthogonal of order $r+1$ with respect to the Laguerre weight $x^{\alpha+r+1}e^{-x}$, that is,

$$\int_0^{+\infty} p(x)Q_{n,r}(x)x^{\alpha+r+1}e^{-x}dx = 0,$$

for every polynomial p with $\deg p \leq n-(r+1)-1$. Therefore, as an immediate consequence we have a *connexion formula* of the form

$$Q_{n,r}(x) = \sum_{j=0}^{r+1} a_{n,r}^j L_{n-j}^{\alpha+r+1}(x)\,, \quad a_{n,r}^0 = 1. \tag{16}$$

In this section, our first effort is devoted to obtain the size of the *connexion coefficients* $a_{n,r}^j$. We introduce a fruitful and new technique which leads us to know the asymptotic behaviour of these coefficients.

Then, we get two asymptotic properties of the Sobolev type orthogonal polynomials $Q_{n,r}$. The first one is the *relative asymptotics*, which shows that the Laguerre–Sobolev type polynomials are very similar to the Laguerre polynomials outside the support of the measure. The second one is the so–called Mehler–Heine type formula which shows how the presence of the masses in the inner product changes the asymptotic behaviour around the origin.

As we will see later, it is worth noticing that the knowledge of the asymptotic behaviour of the connexion coefficients is not enough to obtain both asymptotic properties. Thus, we can assure that the techniques used in the bounded case do not work now.

4.1 Connexion coefficients

Notice that the sequence $\{Q_{n,r}\}_{n\geq 0}$ is quasi–orthogonal of order $r+1$ with respect to the Laguerre weight $x^{\alpha+r+1}e^{-x}$ and therefore we have a *connexion formula* of the form

$$Q_{n,r}(x) = \sum_{j=0}^{r+1} a_{n,r}^j L_{n-j}^{\alpha+r+1}(x)\,.$$

Using this expansion, in the next lemma we obtain a new algebraic expression with a nice structure which relates the derivative of order $k+1$ of the polynomials $Q_{n,r}$ to the derivative of order k.

Lemma 3 *Fixed $r \geq 1$, let $\{Q_{n,r}\}_{n\geq 0}$ be the SMOP with respect to the inner product (9). Then, we have for $0 \leq k \leq n-1$,*

$$\begin{aligned}\frac{Q_{n,r}^{(k+1)}(0)}{(L_n^\alpha)^{(k+1)}(0)} &= \frac{\alpha+k+1}{\alpha+r+k+2}\frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} \\ &+ \frac{\Gamma(\alpha+r+2)}{\Gamma(\alpha+r+k+3)}\sum_{i=1}^{k+1}\binom{k}{i-1}\frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+i+1)}\frac{A_{n,r}^i}{(L_n^\alpha)^{(i)}(0)}\,,\end{aligned} \tag{17}$$

where

$$A_{n,r}^i = \sum_{j=i}^{r+1}\frac{j!}{(j-i)!}\,a_{n,r}^j\,L_{n-j}^{\alpha+r+1}(0), \quad i=0,1,\ldots,r+1\,,$$

and the coefficients $a_{n,r}^j$ are those of (16). By convention, $A_{n,r}^i = 0$, when $i > r+1$. Besides,

$$\lim_n \frac{A_{n,r}^i}{(L_n^\alpha)^{(i)}(0)} = \begin{cases} 0 & \textit{if} \quad 0 \leq i \leq r\,, \\ (r+1)! & \textit{if} \quad i = r+1\,. \end{cases} \tag{18}$$

Proof. Taking derivatives $k+1$ times in (16), evaluating at $x=0$, and using (4) several times we get, for $k \geq 0$,

$$\begin{aligned} Q_{n,r}^{(k+1)}(0) &= \sum_{j=0}^{r+1} a_{n,r}^j (L_{n-j}^{\alpha+r+1})^{(k+1)}(0) = -\sum_{j=0}^{r+1} a_{n,r}^j \frac{n-j-k}{\alpha+r+k+2}(L_{n-j}^{\alpha+r+1})^{(k)}(0) \\ &= -\frac{n-k}{\alpha+r+k+2}Q_{n,r}^{(k)}(0) + \frac{1}{\alpha+r+k+2}\sum_{j=1}^{r+1} j a_{n,r}^j (L_{n-j}^{\alpha+r+1})^{(k)}(0) \\ &= -\frac{n-k}{\alpha+r+k+2}Q_{n,r}^{(k)}(0) \\ &+ \frac{(-1)^k\,\Gamma(\alpha+r+2)}{\Gamma(\alpha+r+k+3)}\sum_{j=1}^{r+1} j a_{n,r}^j \frac{(n-j)!}{(n-j-k)!}L_{n-j}^{\alpha+r+1}(0)\,. \end{aligned}$$

According to formula (5) in page 8 of [16],

$$\frac{(n-j)!}{(n-j-k)!} = k!\sum_{i=0}^{k}(-1)^i\binom{j-1}{i}\binom{n-1-i}{k-i},$$

and therefore

$$\frac{(n-j)!}{(n-j-k)!} = \frac{(j-1)!}{(n-(k+1))!} \sum_{i=1}^{k+1} (-1)^{i-1} \binom{k}{i-1} \frac{(n-i)!}{(j-i)!} \,.$$

Thus, the above expression can be written in the form:

$$\begin{aligned} Q_{n,r}^{(k+1)}(0) &= -\frac{n-k}{\alpha+r+k+2} Q_{n,r}^{(k)}(0) \\ &+ \frac{(-1)^k \Gamma(\alpha+r+2)}{\Gamma(\alpha+r+k+3)} \sum_{i=1}^{k+1} (-1)^{i-1} \binom{k}{i-1} \frac{(n-i)!}{(n-(k+1))!} A_{n,r}^{i} \,, \end{aligned}$$

which leads to

$$\begin{aligned} \frac{Q_{n,r}^{(k+1)}(0)}{(L_n^{\alpha})^{(k+1)}(0)} &= -\frac{n-k}{\alpha+r+k+2} \frac{(L_n^{\alpha})^{(k)}(0)}{(L_n^{\alpha})^{(k+1)}(0)} \frac{Q_{n,r}^{(k)}(0)}{(L_n^{\alpha})^{(k)}(0)} \\ &+ \frac{(-1)^k \Gamma(\alpha+r+2)}{\Gamma(\alpha+r+k+3)} \sum_{i=1}^{k+1} (-1)^{i-1} \binom{k}{i-1} \frac{(n-i)!}{(n-(k+1))!} \frac{(L_n^{\alpha})^{(i)}(0)}{(L_n^{\alpha})^{(k+1)}(0)} \frac{A_{n,r}^{i}}{(L_n^{\alpha})^{(i)}(0)} \\ &= \frac{\alpha+k+1}{\alpha+r+k+2} \frac{Q_{n,r}^{(k)}(0)}{(L_n^{\alpha})^{(k)}(0)} \\ &+ \frac{\Gamma(\alpha+r+2)}{\Gamma(\alpha+r+k+3)} \sum_{i=1}^{k+1} \binom{k}{i-1} \frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+i+1)} \frac{A_{n,r}^{i}}{(L_n^{\alpha})^{(i)}(0)} \,, \end{aligned}$$

where we have used expression (4). So, the first part of the lemma is proved.

To prove (18), it is enough to apply a recursive procedure beginning with $k=0$ in (17) and take into account Lemma 2 (a). $\Box$

The above result allows us to obtain an estimate of the connexion coefficients $a_{n,r}^{j}$. Observe that the following result is the break point with respect to the techniques used in the bounded case because we establish that the coefficients in (16) are unbounded for $j \geq 1$.

Theorem 1 *Let $a_{n,r}^{j}$ be the connexion coefficients which appear in (16). Then, we have*

$$\lim_{n} \frac{a_{n,r}^{j}}{n^j} = \binom{r+1}{j} \,, \quad 0 \leq j \leq r+1 \,.$$

Proof. From (18) and the expression

$$A_{n,r}^{r+1} = (r+1)!\, a_{n,r}^{r+1} L_{n-r-1}^{\alpha+r+1}(0) = \frac{(r+1)!\; a_{n,r}^{r+1}}{n(n-1)\dots(n-r)} (L_n^{\alpha})^{(r+1)}(0)\,,$$

it follows easily

$$\lim_n \frac{a_{n,r}^{r+1}}{n^{r+1}} = 1.$$

A recurrence procedure leads to the result. Indeed, we assume that the result holds for $k+1 \le j \le r+1$ and we will show that it is true for $j = k$. From (18) for $i = k$ we can obtain

$$\lim_n \sum_{j=k}^{r+1} \frac{j!}{k!\,(j-k)!} \frac{a_{n,r}^{j}}{n^{r+1-k}} \frac{L_{n-j}^{\alpha+r+1}(0)}{(L_n^{\alpha})^{(k)}(0)} = 0\,.$$

From (4) and (5), we get

$$\lim_n \sum_{j=k}^{r+1} (-1)^{k-j} \binom{j}{k} \frac{a_{n,r}^{j}}{n^{j}} = 0.$$

From the assumption

$$\lim_n \frac{a_{n,r}^{j}}{n^{j}} = \binom{r+1}{j}\,, \quad k+1 \le j \le r+1,$$

we have

$$\begin{aligned}\lim_n \frac{a_{n,r}^{k}}{n^{k}} &= \sum_{j=k+1}^{r+1} (-1)^{k+1-j} \binom{j}{k} \binom{r+1}{j} \\ &= \binom{r+1}{k} \sum_{j=k+1}^{r+1} (-1)^{k+1-j} \binom{r+1-k}{j-k} = \binom{r+1}{k}. \quad \square\end{aligned}$$

4.2 Relative asymptotics

The relative asymptotics for the Sobolev type orthogonal polynomials with respect to the Laguerre polynomials cannot be deduced as a consequence of the connexion formula (16). Indeed, from this formula we have

$$\frac{Q_{n,r}(x)}{L_n^{\alpha}(x)} = \sum_{j=0}^{r+1} a_{n,r}^{j} \frac{L_{n-j}^{\alpha+r+1}(x)}{L_n^{\alpha}(x)}.$$

Applying Theorem 1, formulas (7) and (8) in the above expression we deduce that each term tends to infinity with the same order but with an alternating sign, that is,

$$a_{n,r}^{j} \frac{L_{n-j}^{\alpha+r+1}(x)}{L_n^{\alpha}(x)} \sim (-1)^j \binom{r+1}{j} \left(\frac{1}{\sqrt{-x}} \right)^{r+1} n^{\frac{r+1}{2}},$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, \infty)$.

It is important to remark that the asymptotic behaviour of the above terms is totally different from the one when we consider the measures in the Nevai class (bounded case) in which every term has a finite limit.

Theorem 2 *Let $\{Q_{n,r}\}_{n\geq 0}$ be the SMOP with respect to the inner product defined by (9). Then*

$$\lim_{n} \frac{Q_{n,r}(x)}{L_n^{\alpha}(x)} = 1,$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, \infty)$.

Proof. From the Fourier expansion of the polynomial $Q_{n,r}$ in terms of the Laguerre polynomials we have

$$\frac{Q_{n,r}(x)}{L_n^{\alpha}(x)} = 1 - \sum_{i=0}^{r} M_i Q_{n,r}^{(i)}(0) \frac{K_{n-1}^{(0,i)}(x,0)}{L_n^{\alpha}(x)}.$$

Substituting Lemma 1 (a) in this formula, we get

$$\frac{Q_{n,r}(x)}{L_n^{\alpha}(x)} = 1 - \sum_{i=0}^{r} \frac{M_i \, i!}{\|L_{n-1}^{\alpha}\|^2 \, x^{i+1} Q_{n,r}^{(i)}(0) P_i(x,0;L_{n-1}^{\alpha})} \left[1 - \frac{P_i(x,0;L_n^{\alpha})}{P_i(x,0;L_{n-1}^{\alpha})} \frac{L_{n-1}^{\alpha}(x)}{L_n^{\alpha}(x)} \right],$$

where $P_i(x,0;f)$ is the i-th Taylor polynomial of f in 0.

Now we analyze each one of the terms of this sum. Since

$$\lim_n \frac{P_i(x,0;L_n^\alpha)}{(L_n^\alpha)^{(i)}(0)} = \frac{x^i}{i!}\,, \tag{19}$$

from (4) and (7), we have, for $i = 0, \ldots, r$,

$$\lim_n \left[1 - \frac{P_i(x,0;L_n^\alpha)}{P_i(x,0;L_{n-1}^\alpha)} \frac{L_{n-1}^\alpha(x)}{L_n^\alpha(x)} \right] = 0,$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$.

Moreover, taking into account (4), (19), and Lemma 2 (a), there exists

$$\lim_n \frac{M_i\, i! Q_{n,r}^{(i)}(0) P_i(x,0;L_{n-1}^\alpha)}{\|L_{n-1}^\alpha\|^2\, x^{i+1}} \in \mathbb{C}\,.$$

Therefore each one of the terms in the sum tends to 0 uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$ and the result follows. $\square$

4.3 Mehler–Heine type formulas

Once we have proved in the previous subsection that both sequences of orthogonal polynomials, $\{Q_{n,r}\}_{n\geq 0}$ and $\{L_n^\alpha\}_{n\geq 0}$, are asymptotically identical on compact subsets of $\mathbb{C} \setminus [0,\infty)$, we establish their differences through Mehler–Heine type formulas which describe the asymptotic behaviour around the origin. First of all, we recall the corresponding formula for the monic Laguerre polynomials, (see [18, Th.8.1.3]):

$$\lim_n \frac{(-1)^n}{n!\, n^\alpha} L_n^\alpha\left(\frac{x}{n+j}\right) = x^{-\alpha/2} J_\alpha(2\sqrt{x}), \tag{20}$$

uniformly on compact subsets of $\mathbb{C}$ and uniformly on $j \in \mathbb{N} \cup \{0\}$, where J_α is the Bessel function of the first kind of order α ($\alpha > -1$), defined by

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n+\alpha}.$$

Now, we want to obtain a similar result for the SMOP $\{Q_{n,r}\}_{n\geq 0}$.

As it occurs in the study of the relative asymptotics, the Mehler–Heine type formulas cannot be deduced as a consequence of the connexion formula (16). Indeed, from this formula we have

$$\frac{(-1)^n}{n!\,n^\alpha} Q_{n,r}\left(\frac{x}{n+j}\right) = \sum_{i=0}^{r+1} a_{n,r}^i \frac{(-1)^n}{n!\,n^\alpha} L_{n-i}^{\alpha+r+1}\left(\frac{x}{n+j}\right).$$

Therefore, applying Theorem 1 and the Mehler–Heine type formula for Laguerre polynomials, we deduce that each term tends to infinity with the same order but with an alternating sign.

Thus, to get the result for $\{Q_{n,r}\}_{n\geq 0}$, we focus on the problem in a different way. We write the Taylor expansion of the polynomial $Q_{n,r}$

$$\frac{(-1)^n}{n!\,n^\alpha} Q_{n,r}\left(\frac{x}{n+j}\right) = \sum_{k=0}^{n} \frac{(-1)^n}{n!\,n^\alpha} \frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} \frac{(L_n^\alpha)^{(k)}(0)}{k!} \frac{x^k}{(n+j)^k}.$$

Then, to calculate the limit, we are going to use the Lebesgue's dominated convergence theorem. For this purpose, we need to find a uniform bound for the ratios $\dfrac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)}$. It is clear that when we take derivatives k times in the expression (16) the connexion coefficients do not change. Then, it could be thought about the possibility to obtain this uniform bound from this formula. But again we come across the same problem, more concretely

$$\frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} = \sum_{i=0}^{r+1} a_{n,r}^i \frac{(L_{n-i}^{\alpha+r+1})^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)},$$

and therefore, from Theorem 1, again each term tends to infinity with the same order, concretely n^{r+1}, but with an alternating sign.

Then, we come back to the useful expression established in Lemma 3 which leads us to obtain a uniform bound for the ratios $\dfrac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)}$.

Lemma 4 *Let $\{Q_{n,r}\}_{n\geq 0}$ be the SMOP with respect to the inner product (9). Then, fixed $r \geq 1$ there exists a positive integer number n_0 such that for all $n \geq n_0$ and for all k with $r+1 \leq k \leq n$, the inequality*

$$\left|\frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)}\right| \leq 2(r+1)\frac{k!}{(k-r)!}(k-(r-1))\frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+r+k+2)},$$

holds. Furthermore, for $r \geq 0$, there exists $n_0 \in \mathbb{N}$ such that,

$$\left|\frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)}\right| \leq 2(r+1), \quad \forall n \geq n_0 \quad 0 \leq k \leq n. \tag{21}$$

Proof. We prove the Lemma using mathematical induction on k, i.e., on the order of the derivative.

Keeping in mind Lemma 2 (a) for $k = r+1$ and Lemma 3, there exists a positive integer number n_0, independent of k, such that for all $n \geq n_0$, the following formulas hold,

$$\left| \frac{Q_{n,r}^{(r+1)}(0)}{(L_n^\alpha)^{(r+1)}(0)} \right| \leq 4(r+1)(r+1)! \frac{\Gamma(\alpha+r+2)}{\Gamma(\alpha+2r+3)}\,, \tag{22}$$

$$\frac{\Gamma(\alpha+r+2)}{\Gamma(\alpha+i+1)} \left| \frac{A_{n,r}^i}{(L_n^\alpha)^{(i)}(0)} \right| \leq 1\,, \quad i = 1, \ldots, r\,, \tag{23}$$

and

$$\left| \frac{A_{n,r}^{r+1}}{(L_n^\alpha)^{(r+1)}(0)} \right| \leq 2(r+1)!\,. \tag{24}$$

Notice that formula (22) is the required bound for $k = r+1$. Now, we assume that the result holds for a fixed k, with $k \geq r+1$, and then we will deduce that it holds for $k+1$.

Taking absolute values in (17) and using induction hypothesis, (23) and (24), we get for $n \geq n_0$ and $k+1 \leq n$

$$\begin{aligned}
&\left| \frac{Q_{n,r}^{(k+1)}(0)}{(L_n^\alpha)^{(k+1)}(0)} \right| \leq \frac{\alpha+k+1}{\alpha+r+k+2} \left| \frac{Q_{n,r}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} \right| \\
&+ \frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+r+k+3)} \left[\sum_{i=1}^{r} \binom{k}{i-1} + \binom{k}{r} 2(r+1)! \right] \\
&\leq \left[2(r+1) \frac{k!\,(k-(r-1))}{(k-r)!} + 2(r+1) \frac{k!}{(k-r)!} + \sum_{i=1}^{r} \binom{k}{i-1} \right] \frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+r+k+3)} \\
&\leq \left[2(r+1)\, \frac{k!}{(k-r)!}\,(k+1-(r-1)) + r \frac{k!}{(k+1-r)!} \right] \frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+r+k+3)} \\
&\leq \left[2(r+1)\, \frac{(k+1)!}{(k-r)!} + \frac{(k+1)!}{(k+1-r)!} \right] \frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+r+k+3)} \\
&\leq 2(r+1)\, \frac{(k+1)!}{(k+1-r)!}\,(k+1-(r-1)) \frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+r+k+3)}\,.
\end{aligned}$$

So, the first part of Lemma is proved. For the second part, using (4), (10) and Lemma 1 (b) we deduce for every $n \geq k$ the explicit expression

$$\frac{Q_{n,0}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} = 1 - \frac{M_0 K_{n-1}(0,0)}{1+M_0 K_{n-1}(0,0)} \frac{\alpha+1}{\alpha+k+1} \frac{n-k}{n}.$$

Then,

$$0 < \frac{Q_{n,0}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} < 1,$$

holds for all k with $0 \leq k \leq n$. According to this fact and the first part of Lemma 4, we have (21). $\Box$

In the next theorem we show how the presence of the masses in the inner product changes the asymptotic behaviour around the origin. This result proves the conjecture posed in [5].

Theorem 3 *Let $\{Q_{n,r}\}_{n\geq 0}$ be the SMOP with respect to the inner product (9). Then,*

$$\lim_n \frac{(-1)^n}{n!\, n^\alpha} Q_{n,r}\left(\frac{x}{n+j}\right) = (-1)^{r+1} x^{-\alpha/2} J_{\alpha+2r+2}(2\sqrt{x}),$$

uniformly on compact subsets of $\mathbb{C}$ and uniformly on $j \in \mathbb{N}\cup\{0\}$.

Proof. Using the Taylor expansion of the polynomial $Q_{n,r}$ we can write

$$\frac{(-1)^n}{n!\, n^\alpha} Q_{n,r}\left(\frac{x}{n+j}\right) = \sum_{k=0}^{n} \frac{(-1)^n}{n!\, n^\alpha} \frac{Q_{n,r}^{(k)}(0)}{k!} \frac{x^k}{(n+j)^k}.$$

To obtain the asymptotic behaviour of the above expression when $n \to \infty$, we are going to use use the Lebesgue's dominated convergence theorem. Indeed, given a compact set $K \subset \mathbb{C}$, from (4), (5), and (21) in Lemma 4 there exists a positive integer number n_0 such that for all $n \geq n_0$, for all $j \geq 0$ and for all $x \in K$,

$$\frac{1}{n!\, n^\alpha}\left|\frac{Q_{n,r}^{(k)}(0)}{k!}\frac{x^k}{(n+j)^k}\right| \leq \frac{\Gamma(\alpha+1)|L_n^\alpha(0)|}{n!\, n^\alpha} \frac{n!}{(n-k)!\,(n+j)^k} \frac{2(r+1)}{\Gamma(\alpha+k+1)} \frac{|x|^k}{k!}$$
$$\leq \frac{4(r+1)}{\Gamma(\alpha+k+1)} \frac{M^k}{k!},$$

for each $k \geq 0$, where M is a positive constant depending on K. As $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha+k+1)} \frac{M^k}{k!}$ converges, the assumptions of the Lebesgue's dominated convergence theorem are satisfied. Then, using Lemma 2 (a), (4), and

(5), we have

$$\lim_n \sum_{k=0}^{n} \frac{(-1)^n}{n!\, n^\alpha} \frac{Q_{n,r}^{(k)}(0)}{k!} \frac{x^k}{(n+j)^k}$$
$$= \sum_{k=r+1}^{\infty} \frac{(-1)^k}{(k-(r+1))!} \frac{1}{\Gamma(\alpha+k+r+2)}\, x^k = (-1)^{r+1} x^{-\alpha/2} J_{\alpha+2r+2}(2\sqrt{x})\,,$$

uniformly on compact subsets of $\mathbb{C}$ and uniformly on $j \in \mathbb{N} \cup \{0\}$. Thus, the result follows. $\square$

In the next corollary we will show a remarkable difference between the zeros of the orthogonal polynomials L_n^α and the ones of $Q_{n,r}$ concerning the convergence acceleration to 0.

Before analyzing this, we recall (see [18]) that the zeros of the Laguerre polynomials are real, simple and they are located in $(0, \infty)$. We denote by $(x_{n,k})_{k=1}^n$ the zeros of L_n^α in an increasing order. It is worth pointing out that they satisfy the interlacing property $0 < x_{n+1,1} < x_{n,1} < x_{n+1,2} < \ldots$, and that $x_{n,k} \underset{n}{\to} 0$ for each fixed k.

Let $(j_{\alpha,k})_{k\geq 1}$ be the positive zeros of the Bessel function J_α in an increasing order. Then, formula (20) and Hurwitz's theorem lead us to

$$n x_{n,k} \underset{n}{\to} j_{\alpha,k}, \quad k \geq 1,$$

and therefore

$$x_{n,k} \sim \frac{C_k}{n}, \quad k \geq 1,$$

where C_k is a positive constant depending on k.

Concerning the zeros of $Q_{n,r}$, standard arguments (see for instance [6]) allow us to establish that $Q_{n,r}$ has at least $n-(r+1)$ zeros with odd multiplicity in $(0, +\infty)$, or equivalently $n-(r+1)$ changes of sign. Moreover, since $M_0 > 0$ and the mass point in the discrete part of the inner product belongs to the boundary of $(0, +\infty)$ then the number of zeros with odd multiplicity is at least $n-r$ (see [2]).

From Theorem 3 and Hurwitz's theorem and taking into account the multiplicity of 0 as a zero of the limit function in Theorem 3 we achieve

Corollary 1 *Let $(\xi_{n,k}^r)_{k=1}^n$ be the zeros of $Q_{n,r}$. Then*

$$n\, \xi_{n,k}^r \underset{n}{\to} 0, \quad 1 \leq k \leq r+1,$$

$$n\,\xi^r_{n,k} \underset{n}{\to} j_{\alpha+2r+2,k-r-1}, \quad k \geq r+2.$$

Remark 1. The presence of the positive masses M_i, $i = 0,\ldots,r$, in the inner product produces a convergence acceleration to 0 of $r+1$ zeros of the polynomials $Q_{n,r}$.

4.4 Generalized Hermite–Sobolev polynomials

As a consequence of the previous results, we are going to establish asymptotic properties for the orthogonal polynomials associated with the following inner product

$$(p,q) = \int_{\mathbb{R}} p(x)q(x)|x|^{2\mu}\, e^{-x^2} dx + \sum_{i=0}^{2r+1} M_i\, p^{(i)}(0)\, q^{(i)}(0), \tag{25}$$

with $\mu > -1/2$ and $M_i > 0$, $\quad i = 0,\ldots,2r+1$. We denote by $S^{\mu}_{n,r}$ their monic orthogonal polynomials.

The polynomials H^{μ}_n orthogonal with respect to the weight $|x|^{2\mu}\, e^{-x^2}$ ($\mu > -1/2$) are called *generalized Hermite polynomials*.

Notice that in this case the polynomials $S^{\mu}_{n,r}$ are symmetric, that is, $S^{\mu}_{n,r}(-x) = (-1)^n\, S^{\mu}_{n,r}(x)$, and because of this symmetry, we can transform this inner product (25) into an inner product like (9) and so we can establish a simple relation between the polynomials $S^{\mu}_{n,r}$ and the polynomials $Q_{n,r}$ considered before. This technique is known as a symmetrization process. In fact, in [6] this process is considered for standard inner products associated with positive measures. The simplest case of this situation is the relation between monic Laguerre polynomials and Hermite polynomials, that is (see [6] or [18]),

$$H_{2n}(x) = L^{-1/2}_n(x^2), \quad H_{2n+1}(x) = xL^{1/2}_n(x^2), \quad n \geq 0.$$

Later in [3] the authors generalize the symmetrization process in the framework of Sobolev type orthogonal polynomials, see Theorem 2 in [3].

As a consequence we have

$$S^{\mu}_{2n,r}(x) = Q^{\mu-1/2}_{n,r}(x^2), \quad S^{\mu}_{2n+1,r}(x) = xQ^{\mu+1/2}_{n,r}(x^2)$$

where $\{Q^{\mu-1/2}_{n,r}\}_{n\geq 0}$ (respectively, $\{Q^{\mu+1/2}_{n,r}\}_{n\geq 0}$) is the SMOP with respect to an inner product like (9) with $\alpha = \mu - 1/2$ (respectively, $\alpha = \mu + 1/2$).

Thus, applying the above Theorem 2 and Theorem 3 in a straightforward way we obtain

Proposition 1 *Let $\{S_{n,r}^{\mu}\}_{n\geq 0}$ be the SMOP with respect to the inner product (25). Then,*

(a)

$$\lim_n \frac{S_{n,r}^{\mu}(x)}{H_n^{\mu}(x)} = 1,$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$.

(b)

$$\lim_n \frac{(-1)^n\sqrt{n}}{n!\,n^{\mu}} S_{2n,r}^{\mu}\left(\frac{x}{2\sqrt{n+j}}\right) = (-1)^{r+1}\left(\frac{x}{2}\right)^{-\mu+1/2} J_{\mu+2r+3/2}(x),$$
$$\lim_n \frac{(-1)^n}{n!\,n^{\mu}} S_{2n+1,r}^{\mu}\left(\frac{x}{2\sqrt{n+j}}\right) = (-1)^{r+1}\left(\frac{x}{2}\right)^{-\mu+1/2} J_{\mu+2r+5/2}(x),$$

uniformly on compact subsets of $\mathbb{C}$ and uniformly on $j\in\mathbb{N}\cup\{0\}$.

Remark 2. These results generalize some of the results in [4] and solve the conjecture posed there.

5 Inner products with holes

In this section, we are concerned with inner products such that in their discrete part at least one of the masses vanishes:

$$(p,q)_{r,s} = (p,q)_r + M_s p^{(s)}(0)q^{(s)}(0), \quad s\geq r+2, \tag{26}$$

where $M_s>0$, and in $(.,.)_r$ all the masses are positive. That is, roughly speaking, there is a "hole" in the discrete part of the inner product $(.,.)_{r,s}$. We denote by $\{T_{n,r,s}\}_{n\geq 0}$ the sequence of monic polynomials orthogonal with respect to the inner product $(.,.)_{r,s}$.

The Fourier expansion of the polynomial $T_{n,r,s}$ in the orthogonal basis $\{Q_{n,r}\}_{n\geq 0}$ gives

$$T_{n,r,s}(x) = Q_{n,r}(x) - \frac{M_s Q_{n,r}^{(s)}(0)}{1+M_s K_{n-1,r}^{(s,s)}(0,0)} K_{n-1,r}^{(0,s)}(x,0)\,. \tag{27}$$

Using similar arguments as in Lemma 2 it can be proved the following

Lemma 5 *Let $\{T_{n,r,s}\}_{n\geq 0}$ be the SMOP with respect to the inner product (26). Then the following statements hold:*

(a) *For either $0 \le k \le r$ or $k = s$,*

$$\frac{T_{n,r,s}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} \sim \frac{C_{r,s,k}}{n^{\alpha+2k+1}},$$

where $C_{r,s,k}$ is a nonzero real number independent of n.

For $k \ge r+1$ and $k \ne s$

$$\lim_n \frac{T_{n,r,s}^{(k)}(0)}{(L_n^\alpha)^{(k)}(0)} = \frac{k!}{(k-(r+1))!} \frac{k-s}{\alpha+s+k+1} \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+r+k+2)}.$$

(b)

$$\lim_n \frac{(T_{n,r,s}T_{n,r,s})_{r,s}}{\|L_n^\alpha\|^2} = 1.$$

The above lemma also allows us to deduce the relative asymptotics for these orthogonal polynomials.

Theorem 4 *Let $\{T_{n,r,s}\}_{n\ge 0}$ be the SMOP with respect to the inner product defined by (26). Then*

$$\lim_n \frac{T_{n,r,s}(x)}{L_n^\alpha(x)} = 1,$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$.

Proof. Proceeding as in Theorem 2 we have

$$\frac{T_{n,r,s}(x)}{L_n^\alpha(x)} = 1 - \sum_{i=0}^{r} M_i T_{n,r,s}^{(i)}(0) \frac{K_{n-1}^{(0,i)}(x,0)}{L_n^\alpha(x)} - M_s T_{n,r,s}^{(s)}(0) \frac{K_{n-1}^{(0,s)}(x,0)}{L_n^\alpha(x)},$$

and in this way, we can prove that each one of the terms in the above sum converges to 0 uniformly on compact subsets of $\mathbb{C} \setminus [0,\infty)$ and the result follows. □

The Mehler–Heine type formula for the polynomials $\{T_{n,r,s}\}_{n\ge 0}$ is also obtained now.

Theorem 5 *Let $\{T_{n,r,s}\}_{n\ge 0}$ be the SMOP with respect to the inner product (26). Then,*

$$\lim_n \frac{(-1)^n}{n!\,n^\alpha} T_{n,r,s}\left(\frac{x}{n+j}\right) = (-1)^{r+1} x^{-\alpha/2} \tag{28}$$

$$\times \left[\frac{-(s-(r+1))}{\alpha+r+s+2} J_{\alpha+2r+2}(2\sqrt{x}) + \sum_{i=2}^{s-r+1} \lambda_i J_{\alpha+2r+2i}(2\sqrt{x})\right],$$

where λ_i are nonzero real numbers. The limit holds uniformly on compact subsets of $\mathbb{C}$ and uniformly on $j \in \mathbb{N} \cup \{0\}$.

Proof. From (27),

$$\begin{aligned} &\frac{(-1)^n}{n!\,n^\alpha} T_{n,r,s}\left(\frac{x}{n+j}\right) \\ &= \frac{(-1)^n}{n!\,n^\alpha} Q_{n,r}\left(\frac{x}{n+j}\right) - \frac{M_s Q_{n,r}^{(s)}(0)}{1+M_s K_{n-1,r}^{(s,s)}(0,0)} \frac{(-1)^n}{n!\,n^\alpha} \sum_{k=0}^{n-1} \frac{K_{n-1,r}^{(k,s)}(0,0)}{k!} \left(\frac{x}{n+j}\right)^k . \end{aligned} \tag{29}$$

To estimate the kernels $K_{n-1,r}^{(k,s)}(0,0)$ we apply Stolz criterion, Lemma 2, (4) and (6), obtaining

$$\begin{aligned} &\lim_n \frac{K_{n-1,r}^{(k,s)}(0,0)}{n^{\alpha+k+s+1}} \\ &= \begin{cases} 0 & \text{if } 0 \le k \le r\,, \\ \frac{k!}{(k-(r+1))!} \frac{s!}{(s-(r+1))!} \frac{(-1)^{k+s}\Gamma(\alpha+1)}{(\alpha+k+s+1)\Gamma(\alpha+k+r+2)\Gamma(\alpha+s+r+2)} & \text{if } k \ge r+1\,. \end{cases} \end{aligned}$$

Moreover, it is not difficult to check that

$$\lim_n \frac{(-1)^n\, n^{s+1}}{n!} \frac{Q_{n,r}^{(s)}(0)}{K_{n-1,r}^{(s,s)}(0,0)} = (-1)^s \frac{(s-(r+1))!}{s!} \frac{(\alpha+2s+1)\Gamma(\alpha+s+r+2)}{\Gamma(\alpha+1)} .$$

According to the two above results, we get the asymptotic behaviour of the coefficients in the sum appearing in (29),

$$\begin{aligned} &\lim_n \frac{(-1)^n}{n!\,n^{\alpha+k}} \frac{M_s Q_{n,r}^{(s)}(0) K_{n-1,r}^{(k,s)}(0,0)}{1+M_s K_{n-1,r}^{(s,s)}(0,0)} \\ &= \begin{cases} 0 & \text{if } 0 \le k \le r\,, \\ \frac{(-1)^k k!}{(k-(r+1))!} \frac{(\alpha+2s+1)}{(\alpha+k+s+1)\Gamma(\alpha+k+r+2)} & \text{if } k \ge r+1\,. \end{cases} \end{aligned}$$

On the other hand, from (21) there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, and for every $k, s \ge 0$ we have

$$\left| K_{n-1,r}^{(k,s)}(0,0) \right| \le 4(r+1)^2 \left| K_{n-1}^{(k,s)}(0,0) \right| .$$

Now, to obtain a bound for the kernels $K_{n-1}^{(k,s)}(0,0)$, we consider the expression which appears in Lemma 1 (b). Then, when $k \geq s-1$, it is easy to check that $j!\,(k+s+1-j)! \geq s!\,(k+1)!$, and $\Gamma(\alpha+j+1)\,\Gamma(\alpha+k+s+2-j) \geq \Gamma(\alpha+s+1)\,\Gamma(\alpha+k+2)$ for $0 \leq j \leq s$. Therefore,

$$\left|K_{n-1}^{(k,s)}(0,0)\right| \leq C_s\, \frac{n^{\alpha+k+s+1}}{\Gamma(\alpha+k+2)}\,.$$

Indeed, given a compact set $K \subset \mathbb{C}$, from (4), (5), and (21), there exists a positive integer number n_0 such that for all $n \geq n_0$, for all $k \geq s-1$, for all $j \geq 0$ and for all $x \in K$,

$$\frac{1}{n^{\alpha+s+1}}\left|\frac{K_{n-1,r}^{(k,s)}(0,0)}{k!}\,\frac{x^k}{(n+j)^k}\right| \leq C_s \frac{4(r+1)^2}{\Gamma(\alpha+k+2)}\,\frac{M^k}{k!},$$

where M is a positive constant depending on K. As $\sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha+k+2)}\frac{M^k}{k!}$ converges, we can apply of the Lebesgue's dominated convergence in the last term of (29). Then, using Theorem 3, we obtain

$$\lim_n \frac{(-1)^n}{n!\,n^{\alpha}} T_{n,r,s}\left(\frac{x}{n+j}\right) = (-1)^{r+1}x^{-\alpha/2}J_{\alpha+2r+2}(2\sqrt{x}) \tag{30}$$
$$- \sum_{k=r+1}^{\infty} \frac{(-1)^k(\alpha+2s+1)}{(\alpha+k+s+1)\Gamma(\alpha+k+r+2)}\,\frac{x^k}{(k-(r+1))!}.$$

If we write $s = r+1+h$ with $h \geq 1$, then the above series is read as

$$(-1)^{r+1}x^{r+1}(\alpha+2r+2h+3)\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k+2r+h+3)}{\Gamma(\alpha+k+2r+3)\Gamma(\alpha+k+2r+h+4)}\,\frac{(-1)^k x^k}{k!}.$$

Observe that $\Gamma(\alpha+k+2r+h+3)/\Gamma(\alpha+k+2r+3)$ is a polynomial in k of degree h (the number of holes) and so we can write

$$\frac{\Gamma(\alpha+k+2r+h+3)}{\Gamma(\alpha+k+2r+3)} = \frac{\Gamma(\alpha+2r+h+3)}{\Gamma(\alpha+2r+3)} + \sum_{l=1}^{h}\beta_l k^l$$

where β_l, $l = 1,\ldots,h$ are positive coefficients. Thus, the above series can be expressed as

$$\frac{\Gamma(\alpha+2r+h+3)}{\Gamma(\alpha+2r+3)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha+k+2r+h+4)}\,\frac{(-1)^k x^k}{k!}$$
$$+\sum_{l=1}^{h}\beta_l\sum_{k=0}^{\infty}\frac{k^l}{\Gamma(\alpha+k+2r+h+4)}\,\frac{(-1)^k x^k}{k!}.$$

For the first one, using the recurrence relation repeatedly (see [18]),

$$J_{\alpha-1}(2\sqrt{x}) + J_{\alpha+1}(2\sqrt{x}) = \alpha\, x^{-\frac{1}{2}}\, J_\alpha(2\sqrt{x}),$$

we get

$$\begin{aligned}
&\frac{\Gamma(\alpha+2r+h+3)}{\Gamma(\alpha+2r+3)} \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha+k+2r+h+4)} \frac{(-1)^k x^k}{k!} \\
&= \frac{\Gamma(\alpha+2r+h+3)}{\Gamma(\alpha+2r+3)}\, x^{-\frac{\alpha+2r+h+3}{2}}\, J_{\alpha+2r+h+3}(2\sqrt{x}) \\
&= x^{-\frac{\alpha+2r+2}{2}} \left[\frac{1}{\alpha+2r+h+3} J_{\alpha+2r+2}(2\sqrt{x}) + \sum_{i=2}^{h+2} \mu_i J_{\alpha+2r+2i}(2\sqrt{x}) \right],
\end{aligned}$$

where μ_i are real numbers which can be computed explicitly.

Moreover, for the remaining series, using the same arguments it can be seen that each one of the terms can be written as a combination of Bessel functions of order bigger than $\alpha+2r+2$. More precisely, for $l = 1, \ldots, h$

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{k^l}{\Gamma(\alpha+k+2r+h+4)} \frac{(-1)^k x^k}{k!} = -x \sum_{k=0}^{\infty} \frac{(k+1)^{l-1}}{\Gamma(\alpha+k+2r+h+5)} \frac{(-1)^k x^k}{k!} \\
&= x^{-\frac{\alpha+2r+2}{2}} \left[-\frac{\Gamma(\alpha+2r+5)}{\Gamma(\alpha+2r+h+5)} J_{\alpha+2r+4}(2\sqrt{x}) + \sum_{i=3}^{h+2} \mu_i^* J_{\alpha+2r+2i}(2\sqrt{x}) \right],
\end{aligned}$$

where μ_i^* are again real numbers which can be computed explicitly.

Finally, taking these results into account in (30) we achieve

$$\begin{aligned}
\lim_n \frac{(-1)^n}{n!\, n^\alpha} T_{n,r,s}\left(\frac{x}{n+j}\right) &= (-1)^{r+1} x^{-\alpha/2} \left[1 - \frac{\alpha+2r+2h+3}{\alpha+2r+h+3} \right] J_{\alpha+2r+2}(2\sqrt{x}) \\
&\quad + (-1)^{r+1} x^{-\alpha/2} \sum_{i=2}^{h+2} \lambda_i J_{\alpha+2r+2i}(2\sqrt{x}),
\end{aligned}$$

and the proof is concluded. □

We can also use the techniques developed in this section to obtain asymptotics results in other similar frameworks. For example, very recently in [8]

the authors consider the inner product

$$
\begin{aligned}
(p,q)_* &= \frac{1}{\Gamma(\alpha+1)}\int_0^\infty p(x)q(x)\,x^\alpha e^{-x}\,dx + M_s p^{(s)}(0)q^{(s)}(0) \\
&= (p,q) + M_s p^{(s)}(0)q^{(s)}(0),
\end{aligned}
\tag{31}
$$

where $s \geq 1$. Notice that the role played by $(p,q)_r$ in the inner product (26) is now played by (p,q) in (31). Therefore, if we proceed as in this section, we improve the asymptotic results appearing in [8].

Moreover, for the particular case $s = r+2$ in the inner product (26), i.e., when there is a hole of "length one", the result established in the above theorem generalize the one obtained in [5]. In fact, handling the right–hand side of the expression (30) and using the recurrence relation of the Bessel functions we obtain:

$$
\begin{aligned}
&\lim_n \frac{(-1)^n}{n!\,n^\alpha} T_{n,r,r+2}\left(\frac{x}{n+j}\right) = (-1)^{r+1} x^{-\alpha/2} \\
&\times \left[\frac{-1}{\alpha+2r+4} J_{\alpha+2r+2}(2\sqrt{x}) - J_{\alpha+2r+4}(2\sqrt{x}) + \frac{-1}{\alpha+2r+4} J_{\alpha+2r+6}(2\sqrt{x})\right].
\end{aligned}
$$

On the other hand, as a consequence of the above theorem we present the situation about the acceleration of the convergence towards the origin of the zeros of the polynomials $T_{n,r,s}$. The quasi–orthogonality of order $s+1$ of the sequence $\{T_{n,r,s}\}_{n\geq 0}$ with respect to the positive measure $x^{\alpha+s+1}e^{-x}$ assures that $T_{n,r,s}$ has at least $n-(s+1)$ changes of sign in $(0,+\infty)$. However, in [2] the authors proved that the number of zeros in $(0,+\infty)$ does not depend on the order of the derivatives but on the number of terms in the discrete part of the inner product. So, $T_{n,r,s}$ has at least $n-(r+1)$ zeros with odd multiplicity in $(0,+\infty)$.

From Theorem 5 and Hurwitz's theorem and taking into account that $x=0$ is a zero of multiplicity $r+1$ of the limit function in (28) we achieve the following result:

Corollary 2 *Let $(\zeta_{n,k}^{r,s})_{k=1}^n$ be the zeros of $T_{n,r,s}$. Then*

$$
n\,\zeta_{n,k}^{r,s} \underset{n}{\to} 0, \quad 1 \leq k \leq r+1,
$$

$$
n\,\zeta_{n,k}^{r,s} \underset{n}{\to} j_{\alpha+2r+2,k-r-1}, \quad k \geq r+2.
$$

In the next remark, we compare the previous result with the corresponding one of Corollary 1.

Remark 3. We want to highlight that the convergence acceleration to 0 of the zeros of the polynomials $Q_{n,r}$ and $T_{n,r,s}$ is the same. That is, the addition of a mass M_s *after a hole* in the inner product does not affect the convergence acceleration to 0.

As we have explained in the previous section, using a symmetrization process, we can obtain the relative asymptotics and the Mehler–Heine type formulas for generalized Hermite–Sobolev polynomials with holes in the discrete part of the inner product.

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